Electromagnetic Wave Propagation in a Rectangular Waveguide with Sinusoidally Varying Width

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Abstract—**Wave propagation along a rectangular waveguide with slowly varying width has been investigated with the help of field theory and approximate circuit theory. In the field theory approach, two different methods of analysis have been attempted. Many properties of the modulated periodic structure, e.g., the frequency dependence of the propagation constant, group and phase velocities, and the electric field axial variation for the fundamental space harmonic and its filter-like property have been investigated. The magnetic field lines on the *H*-plane for a typical case exhibit an expected configuration. Experimental results show close agreement with analysis. It is concluded that this structure supports the fast fundamental space harmonic.**

I. Introduction

STUDY OF THE electromagnetic properties of periodic structures has been a subject of considerable interest. Investigations have been made of nonreciprocal periodic structures [1], [2] and periodically loaded lines [3]. Zucker [4] and Harvey [5] have considered different types of open and closed periodic structures and have presented a comprehensive bibliography.

In the case of distributed loading, the periodicity may be generated by producing a gradual change in any one or more of the electrical parameters of an otherwise uniform transmission line. As an example of this [6], [7], a uniform hollow waveguide may be filled with a dielectric material whose permittivity changes smoothly and periodically with longitudinal distance. Wave propagation through a structure with sinusoidally modulated reactance walls has similar properties [8]. The same idea has been extended [9]–[11] to the problem where the transverse cross section of a waveguide is made a periodic function of the axial distance. An approximate analysis in respect of field solutions within a circular waveguide of periodically varying cross section has been made by James and Walker [9]. The present paper gives the results of the theoretical and experimental investigations on wave propagation through a rectangular waveguide having sinusoidally varying width along the axial direction.

Considerable insight into the nature of the wave propagation within a periodic structure is obtained from its ω–β diagram. With this end in view, the analysis has been directed to obtain the nature of variation of the phase constant with frequency. Three methods of analysis have been attempted to determine the propagation constant for the fundamental space harmonic. In addition, the expressions for different modal field components have also been derived in this paper.

The method of analysis starts with an appropriate coordinate transformation to make the wave equation separable, thereby allowing the field solutions to be readily obtained. In the separated form, the wave equation involving the longitudinal axis as an independent variable becomes a second-order differential equation with periodically varying coefficients. In the first method, this equation by suitable transformation is reduced to the Hill's equation. The phase shift per period is then evaluated from the characteristic exponent value resulting from this equation. The second method is concerned with the numerical evaluation of the solution of the aforesaid differential equation at different axial points assuming appropriate initial conditions. The exact phase shift as well as the field amplitude in a periodic cell is obtained from the knowledge of the solution at the termination of a period.

In the third method, a circuit theory approach has been resorted to. Here the sinusoidally modulated waveguide is approximated by a series of small sections of rectangular waveguide of varying widths, each carrying a dominant mode. From the knowledge of the wave amplitude transmission matrix of each waveguide section, it is possible to obtain the phase shift of one complete periodic cell.

The analysis presented also yields the field configurations inside each cell of the periodic structure. A typical sample of the magnetic field configuration has been included.

The experimental program undertaken in this study gives the ω–β diagram and the axial variation of the field amplitude in each cell. Results obtained by the three different methods show excellent agreement with experimental data.

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II. Coordinate Transformation

The geometry of the periodic structure is shown in Fig. 1. For the purpose of solving the wave equation, it is necessary to employ a suitable coordinate transformation.

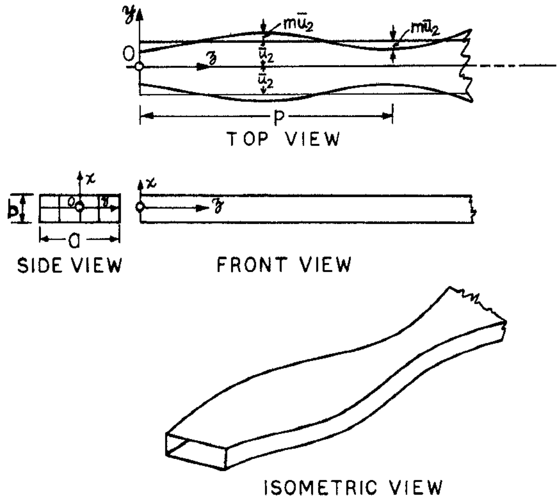

Fig. 1. Geometry of the periodic structure.

The required transformation relating the Cartesian coordinates with the new system of coordinates u_i $(i=1,2,3)$ matching the geometry of the periodic structure is given as

$$\begin{aligned} u_1 &= x \\ u_2 &= y/[1+mf(2\pi z/p)] \\ u_3 &= z+\Delta(y,z) \end{aligned} \tag{1}$$

where $f(2\pi z/p)$ is a periodic function of z with period p, m is the depth of modulation $0<m<1$, and $|f(2\pi z/p)| \leqslant 1$. The undefined function $\Delta(y,z)$ becomes small in magnitude when a slow axial variation of the width is assumed. In the Taylor's series expansions of $f(2\pi z/p)$ and of $\Delta(y,z)$ in z about u_3 higher order terms of $\Delta(y,z)$ may then be neglected; further, the following conditions

$$|(2\pi/p)mf'(2\pi u_3/p)\Delta(y,z)| \ll [1+mf(2\pi u_3/p)] \tag{2a}$$

$$\left|\frac{\partial\Delta(y,u_3)}{\partial u_3}\right| \ll 1 \tag{2b}$$

are assumed to be satisfied. The orthogonality requirements of u_i system of coordinates and the use of (2(b)) gives the derivative of $\Delta(y,u_3)$ with respect to u_2. Integrating this, the undefined function $\Delta(y,u_3)$ is determined as

$$\Delta(y,u_3) \approx (2\pi/p)\frac{1}{2}u_2^2 mf'(2\pi u_3/p) \cdot [1+mf(2\pi u_3/p)]. \tag{3}$$

It can now be shown [12] that the wave equation in the u_i system of coordinates is separable if

$$1/2[(2\pi/p)u_2 mf'(2\pi u_3/p)]^2 \ll 1.$$

Then making use of (3) in (2b), and utilizing the above condition for separability, the ultimate constraint on the physical parameters is

$$\left|\frac{1}{2}(2\pi/p)^2 u_2^2 mf''(2\pi u_3/p) \cdot [1+mf(2\pi u_3/p)]\right| \ll 1. \tag{4}$$

The worst possible constraint will be at a point where (4) becomes maximum and may be defined by a parameter

$$c=(2\pi\bar{u}_2/p)^2 m(1+m) \ll 1. \tag{5}$$

It may be stated here that, though several waveguide structures with different values of parameters have been studied, the results of only two of them having two different sets of parameter values corresponding to $c=0.063$ ($p=10$ cm, $m=0.2$, $\bar{u}_2=1.143$ cm) in set 1, and $c=0.207$ ($p=10$ cm, $m=0.3$, $\bar{u}_2=1.633$ cm) in set 2 have been presented in this paper.

Conditions as given in (2a), (2b), and (4) are all physically realizable and simplify the transformation indicated in (1). Some of the useful expressions required for subsequent analysis are given below.

The Lamé coefficients for the u_i coordinates are

$$\begin{aligned} h_1 &= 1 \\ h_2 &= 1+mf(2\pi u_3/p) \\ h_3 &\approx 1. \end{aligned} \tag{6}$$

The relations between the unit vectors of the two systems of coordinates are

$$\begin{aligned} \bar{i}_1 &= \bar{i}_x \\ \bar{i}_2 &= \bar{i}_y - \bar{i}_z[(2\pi/p)u_2 mf'(2\pi u_3/p)] \\ \bar{i}_3 &= \bar{i}_y[(2\pi/p)u_2 mf'(2\pi u_3/p)] + \bar{i}_z \end{aligned} \tag{7}$$

and the Stäckel determinant is

$$S=\begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1/[1+mf(2\pi u_3/p)]^2 & 0 \end{vmatrix}. \tag{8}$$

Finally, the simplified relations between the two coordinate systems

$$\begin{aligned} x &= u_1 \\ y &= u_2[1+mf(2\pi u_3/p)] \\ z &= u_3 - \Delta(y,u_3). \end{aligned} \tag{9}$$

III. Wave Equation

It is now possible to solve the wave equation in the u_i coordinate system as the requirements for orthogonality and separability are satisfied. The wave equation $\nabla^2\psi + k^2\psi = 0$ expressed in terms of the u_i coordinates takes the general form [12]

$$\frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\psi}{\partial u_i}\right)+k^2\psi=0 \tag{10}$$

where $k^2=\omega^2\mu\epsilon$, ψ is the wave function, and h_i's are the Lamé's coefficients given in (6).

Letting $\psi = U_1(u_1)U_2(u_2)U_3(u_3)$, the wave equation (10) is separated into the following three ordinary differential equations:

$$\frac{d^2U_1}{du_1^2} + K_1^2 U_1 = 0 \tag{11a}$$

$$\frac{d^2U_2}{du_2^2} + K_2^2 U_2 = 0 \tag{11b}$$

$$\frac{d^2U_3}{du_3^2} + \frac{(2\pi/p)mf'(2\pi u_3/p)}{1+mf(2\pi u_3/p)}\frac{dU_3}{du_3} + \left[k^2 - K_1^2 - \frac{K_2^2}{\{1+mf(2\pi u_3/p)\}^2}\right]U_3 = 0. \tag{11c}$$

The methods of solution of the last of the above three equations are discussed in Section V.

IV. Field Solution

The electric and magnetic fields within the periodic structure may exist in the form of various TE and TM modes. It can be shown that in the u_i coordinate system, the solution of the wave equation is not directly obtainable for TE to z or TM to z modes, because $(\nabla^2\bar{F})_z \neq \nabla^2 F_z$, where F_z is the z component of electric vector potential. It is, however, found that the above condition is well satisfied by TE to u_1 and TM to u_1 modes, and the wave equation is readily solvable for these hybrid modes only.

In the u_i coordinate system, the field components for TE to u_1 are obtained by assuming the electric vector potential $\bar{F} = \bar{i}_x\psi^f$ and the magnetic vector potential $\bar{A} = \bar{0}$, and using them in the expressions for the electric field $\bar{E}$ and the magnetic field $\bar{H}$ expressed in terms of $\bar{F}$ and $\bar{A}$ [13]. The field components for TM to u_1 are similarly obtained by taking $\bar{A} = \bar{i}_x\psi^a$ and $\bar{F} = 0$. ψ^f and ψ^a are the wave functions of respective modes and solutions of corresponding scalar Helmholtz equations. Thus the field components are, for TE to u_1 modes

$$E_1 = 0 \qquad H_1 = \frac{1}{j\omega\mu}\left[\omega^2\mu\epsilon + \frac{\partial^2}{\partial u_1^2}\right]\psi^f$$

$$E_2 = -\frac{\partial\psi^f}{\partial u_3} \qquad H_2 = \frac{1}{j\omega\mu}\frac{1}{h_2}\frac{\partial^2\psi^f}{\partial u_1\partial u_2}$$

$$E_3 = \frac{1}{h_2}\frac{\partial\psi^f}{\partial u_2} \qquad H_3 = \frac{1}{j\omega\mu}\frac{\partial^2\psi^f}{\partial u_1\partial u_3} \tag{12a}$$

and for TM to u_1 modes

$$E_1 = \frac{1}{j\omega\epsilon}\left[\omega^2\mu\epsilon + \frac{\partial^2}{\partial u_1^2}\right]\psi^a \qquad H_1 = 0$$

$$E_2 = \frac{1}{j\omega\epsilon}\frac{1}{h_2}\frac{\partial^2\psi^a}{\partial u_1\partial u_2} \qquad H_2 = \frac{\partial\psi^a}{\partial u_3}$$

$$E_3 = \frac{1}{j\omega\epsilon}\frac{\partial^2\psi^a}{\partial u_1\partial u_3} \qquad H_3 = -\frac{1}{h_2}\frac{\partial\psi^a}{\partial u_2}. \tag{12b}$$

Since this structure would be fed by a standard rectangular waveguide, it will be excited with a field that does not vary in x dimension, i.e., $\partial/\partial u_1 = 0$. Then the field components given by (12b) belong to the type of TE to z and those of (12a) to TM to z.

Thus the field components for such TE to z mode are

$$E_1 = -j\omega\mu\psi^a \qquad H_2 = \frac{\partial\psi^a}{\partial u_3}$$

$$E_2 = E_3 = H_1 = 0 \qquad H_3 = -\frac{1}{h_2}\frac{\partial\psi^a}{\partial u_2} \tag{13}$$

where $\psi^a = U_2(u_2)U_3(u_3)$ is the solution obtained from (11). The boundary condition requires that the tangential field $E_1 = 0$ on the conducting walls defined by $u_2 = \pm\bar{u}_2$. Evidently, ψ^a should have two possible types of solution, e.g., 1) symmetric solution about $u_2 = 0$ for which $\psi^a = \cos(K_2u_2)U_3$ with $K_2 = (2n+1)\pi/2\bar{u}_2$, and 2) skew symmetric solution about $u_2 = 0$, where $\psi^a = \sin(K_2u_2)U_3$ with $K_2 = n\pi/\bar{u}_2$. Only the symmetric solution for the wave function will be used in the subsequent discussion.

To express the field components in the Cartesian coordinates (x,y,z), it is necessary to transform the field solution shown in (13) with the use of (7). This yields

$$E_x = -j\omega\mu\cos(K_2u_2)U_3$$

$$E_y = E_z = H_x = 0$$

$$H_y = \cos(K_2u_2)\frac{dU_3}{du_3} + (2\pi/p)u_2 mf'(2\pi u_3/p) \cdot (K_2/h_2)\sin(K_2u_2)U_3$$

$$H_z = (K_2/h_2)\sin(K_2u_2)U_3 - (2\pi/p)u_2 mf'(2\pi u_3/p) \cdot \cos(K_2u_2)\frac{dU_3}{du_3}. \tag{14}$$

For the limiting case where $m = 0$, u_2 tends to y, u_3 to z, and $\bar{u}_2$ to $a/2$, the field expressions of (14) take the usual form.

V. Determination of Propagation Constant

Now the central problem is to solve the differential equation given in (11c). The propagation characteristics of the structure will be revealed when the characteristic exponent of the equation is determined. In this analysis, the subscript 3 is dropped, and U_3 and u_3 are written as U and u, respectively.

For a sinusoidally modulated waveguide $f(2\pi u/p) = -\cos(2\pi u/p)$ (Fig. 1). Using (6), (11c) becomes

$$\frac{d^2U}{du^2}+\frac{(2\pi/p)m\sin(2\pi u/p)}{1-m\cos(2\pi u/p)}\frac{dU}{du}+\left[k^2-\frac{K_2^2}{\{1-m\cos(2\pi u/p)\}^2}\right]U=0. \quad (15)$$

A. Hill's Analysis

Writing $P(u)$ for the coefficient of dU/du in (15) and using the transformations $U=V\exp\{-\frac{1}{2}\int P(u)\,du\}$ and $u=px/\pi$, (15) reduces to the Hill's equation

$$\frac{d^2V}{dx^2}+R(x)V=0 \quad (16)$$

where

$$R(x)=(P/\pi)^2\left[k^2-\frac{K_2^2}{(1-m\cos 2x)^2}\right]. \quad (17)$$

Being an even periodic function of x, $R(x)$ may be expanded into a Fourier series

$$R(x)=\theta_0+2\sum_{r=1}^{\infty}\theta_{2r}\cos(2rx). \quad (18)$$

Let the solution of (16) be written as $V=\phi(x)\exp(\gamma x)$, where the characteristic exponent γ (propagation constant) is in general a complex quantity. The characteristic exponents are obtained by setting the Hill's determinant $\Delta(j\gamma)$ equal to zero. That is

$$\Delta(j\gamma)=\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \dfrac{(j\gamma+2)^2-\theta_0}{2^2-\theta_0} & \dfrac{-\theta_2}{2^2-\theta_0} & \dfrac{-\theta_4}{2^2-\theta_0} & \cdots \\ \cdots & \dfrac{-\theta_2}{0^2-\theta_0} & \dfrac{(j\gamma)^2-\theta_0}{0^2-\theta_0} & \dfrac{-\theta_2}{0^2-\theta_0} & \cdots \\ \cdots & \dfrac{-\theta_4}{2^2-\theta_0} & \dfrac{-\theta_2}{2^2-\theta_0} & \dfrac{(j\gamma-2)^2-\theta_0}{2^2-\theta_0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}=0. \quad (19)$$

It immediately follows that [14]

$$\sin^2(j\gamma\pi/2)=\Delta(0)\sin^2\left(\sqrt{\theta_0}\,\pi/2\right)$$
$$\cosh(\pi\gamma)=1-2\Delta(0)\sin^2\left(\sqrt{\theta_0}\,\pi/2\right). \quad (20)$$

Now comparison of (17) and (18) yields

$$\theta_0=(p/\pi)^2\left[k^2-K_2^2\{1+(3/2)m^2+(15/8)m^4+\cdots\}\right]$$
$$\theta_2=-0.5(p/\pi)^2K_2^2\left[2m+3m^3+(60/16)m^5+\cdots\right]$$
$$\theta_4=-0.5(p/\pi)^2K_2^2\left[(3/2)m^2+(5/2)m^4+(105/32)m^6+\cdots\right]$$
$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots \quad (21)$$
$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

and so on (numerically worked out up to θ_{26}).

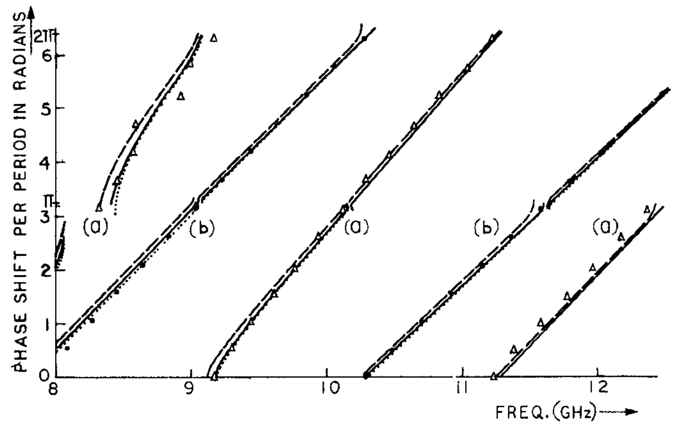

Fig. 2. Plot of the phase shift per period with frequency for two typical periodic structures: (a) $c=0.063$ (set 1), (b) $c=0.207$ (set 2) using 1) Hill's analysis (______), 2) numerical analysis (······), 3) stepped waveguide analysis (_ _ _ _ _), and 4) experimental results for (a) (△△△) and for (b) (●●●). Note: Results by Hill's and numerical analysis become identical for case (a) beyond about 10 GHz and for case (b) beyond about 12 GHz.

From (21), the application of Cauchy's ratio test shows that

$$\sum_{0}^{\infty}\theta_{2r}=(p/\pi)^2\left[k^2-K_2^2(1+2m+3m^2+4m^3+5m^4+6m^5+7m^6+8m^7+\cdots)\right]$$

converges absolutely, thereby securing the convergence of the Hill's determinant [14]. The characteristic exponent γ may be evaluated by using either of the equations of (20) in which $\Delta(0)$ is obtained by substituting $j\gamma=0$ in (19). The infinite determinant $\Delta(0)$, being convergent, is determined by reducing its size to a finite but large order so that only the significant elements around its center are retained. In a dissipationless device, the value of the phase shift per period is obtained by substituting $\gamma=j\beta_0$ in (20). The dependence of the phase shift per period $\beta_0 p$ on frequency as obtained theoretically through this analysis has been given in Fig. 2, which shows pass and stop band characteristics typical of a periodic structure.

B. Numerical Analysis

The general form of (16) may be written as

$$\frac{d^2U}{du^2}+P(u)\frac{dU}{du}+Q(u)U=0. \quad (22)$$

According to Floquet's theorem [14], for a lossless structure, the solution of (22) will have the format $U=A(u)\exp(\pm j\beta_0 u)$. Being a periodic function of u with period p, $A(u)$ may be expanded in a Fourier series. Thus

$$U=\sum_{n=-\infty}^{n=\infty}a_n\exp(\pm j\beta_n u) \quad (23)$$

where $\beta_n=\beta_0+2\pi n/p$. β_n is the propagation constant for the nth space harmonic, while β_0 is the same for the fundamental. For an infinite periodic waveguide excited

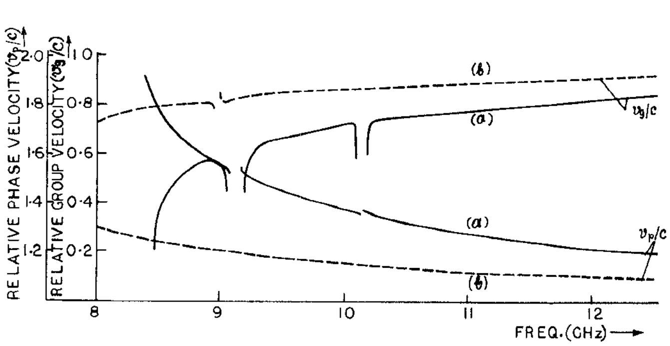

Fig. 3. Dependence of the phase and group velocities (theoretical) of the fundamental space harmonic on frequency for the periodic structures with (a) $c=0.063$ (set 1) (______) and (b) $c=0.207$ (set 2) (_ _ _ _ _).

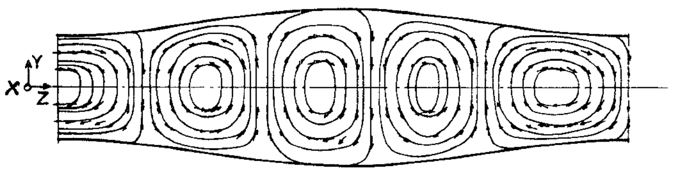

Fig. 4. Magnetic lines of force for the TE_{01} mode in a periodic structure with $c=0.063$ (set 1) at frequency 9.6 GHz (pass band). Arrow indicate tangents to the lines of force.

by a time-harmonic signal the standing-wave field amplitude is

$$U(u)=\sum_{n=-\infty}^{n=\infty} 2a_n \cos(\beta_n u). \quad (24)$$

Assuming the initial condition that $U=1$ and $dU/du=0$ at $u=0$, it can be seen from (24) that $1=\Sigma_{-\infty}^{\infty} 2a_n = 2[\cdots + a_{-3}+a_{-2}+a_{-1}+a_0+a_1+a_2+a_3+\cdots]$. Again, if $U=U(p)$ at $u=p$ it follows from (24) that $\cos(\beta_0 p)=U(p)$. Therefore,

$$\beta_0 p=\cos^{-1}[U(p)]=\cos^{-1}[U(p)]+2\pi r \quad (25)$$

where r is an integer including zero. It may be noted that (24) with the above mentioned boundary conditions belongs to the Sturm–Liouville class.

Equation (15) has been numerically solved using the Runge–Kutta method from which the plot of U (proportional to electric field in the TE to z mode) versus u has been obtained. From the plot of U versus u, the phase shift per period ($\beta_0 p$) has been calculated using (25). The value of r may also be obtained from the plot itself. The propagation constants for other space harmonics may be calculated using the relation $\beta_n=\beta_0+2\pi n/p$.

The variation of the phase shift per period with frequency has been calculated for a range of values of c. The calculated results for only two typical values of c are shown in Fig. 2. The corresponding phase and the group velocities have been found and are given in Fig. 3.

For a particular case, the calculated value of U may be substituted in (14) to yield H_y and H_z, from which the locus of the magnetic lines of force may be drawn. Such magnetic lines, whose slopes (indicated by arrows) were obtained by using a digital computer, have been plotted in Fig. 4 for a typical structure supporting the TE_{01} mode.

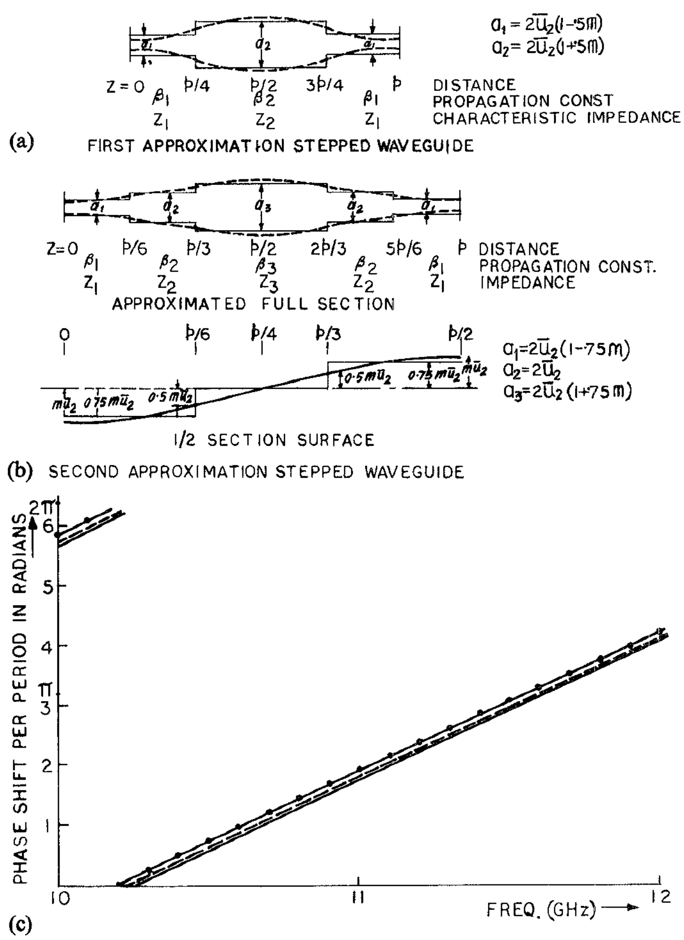

Fig. 5. Stepped waveguide representation of the periodic structure: (a) first-order approximation, (b) second-order approximation, and (c) calculated results of approximate methods for a typical value of $c=0.207$ (set 2); first approx. (•••), second approx. (_ _ _ _ _ _), and third approx. (______).

VI. Stepped Waveguide Analysis

In the next method, the circuit concept of the waveguide discontinuity has been utilized. The restriction given in (5) is no longer important for the subsequent analysis. In this method, the modulated width of the periodic structure is approximated by small symmetrical sections of uniform rectangular waveguides of different widths placed in tandem (Fig. 5). The overall wave amplitude transmission matrix for one complete period of the periodic structure is obtained by multiplication of transmission matrices of individual waveguide sections. The phase shift per period of the structure is determined by using the formula $\cos(\beta_0 p)=(A_{11}+A_{22})/2$, where A_{11} and A_{22} are the two diagonal elements of the overall transmission matrix [16].

In the first-order approximation, one periodic cell has been considered to be represented by waveguides of two different widths as shown in Fig. 5(a). From the overall transmission matrix, the phase shift per period is obtained from

$$\cos(\beta_0 p) = \frac{(Z_1+Z_2)^2\cos 2(\delta_1+\delta_2) - (Z_2-Z_1)^2\cos 2(\delta_1-\delta_2)}{4Z_1Z_2} \tag{26}$$

where

$$\delta_1=\beta_1 p/4=\left[k^2-(\pi/a_1)^2\right]^{1/2}p/4, \qquad Z_1=k\eta/\beta_1, \qquad \eta=\sqrt{\mu/\epsilon}$$

$$\delta_2=\beta_2 p/4=\left[k^2-(\pi/a_2)^2\right]^{1/2}p/4, \qquad Z_2=k\eta/\beta_2.$$

Equation (26) may be further reduced to

$$\cos(\beta_0 p)=(1/4)\left[\left[\sqrt{\frac{\beta_2}{\beta_1}}+\sqrt{\frac{\beta_1}{\beta_2}}\right]^2 \cdot\cos(\beta_1+\beta_2)p/2 - \left[\sqrt{\frac{\beta_1}{\beta_2}}-\sqrt{\frac{\beta_2}{\beta_1}}\right]^2\cos(\beta_1-\beta_2)p/2\right]. \tag{27}$$

In the second-order approximation, each periodic cell is approximated by five uniform rectangular waveguides of three different widths as shown in Fig. 5(b). Following the same procedure, the phase shift may be obtained from

$$\begin{aligned}\cos(\beta_0 p)=\frac{1}{16\beta_1\beta_2^2\beta_3}\Big[&(\beta_1+\beta_2)^2(\beta_2+\beta_3)^2 \cdot\cos(\beta_1+\beta_2+\beta_3)p/3\\ &+(\beta_1-\beta_2)^2(\beta_2-\beta_3)^2\cos(\beta_1-\beta_2+\beta_3)p/3\\ &-(\beta_1+\beta_2)^2(\beta_2-\beta_3)^2\cos(\beta_1+\beta_2-\beta_3)p/3\\ &-(\beta_1-\beta_2)^2(\beta_2+\beta_3)^2\cos(\beta_1-\beta_2-\beta_3)p/3\\ &+2(\beta_2^2-\beta_1^2)(\beta_3^2-\beta_2^2)\cos(\beta_1+\beta_3)p/3\\ &+2(\beta_1^2-\beta_2^2)(\beta_3^2-\beta_2^2)\cos(\beta_1-\beta_3)p/3\Big].\end{aligned} \tag{28}$$

Seven waveguide sections of four different widths give a third-order approximation from which the phase shift per period may be calculated as before. For the purpose of comparison, the calculated values of $\beta_0 p$ for $c=0.063$ obtained for the above three approximations are shown in Fig. 5(c). The computed values of the phase shift per period using the third-order approximation have also been shown in Fig. 2 for two values of c. In this analysis, the basic assumptions made are the following. 1) only the dominant mode is allowed in each of the waveguide sections and, 2) junction susceptances that are formed at the steps have been neglected.

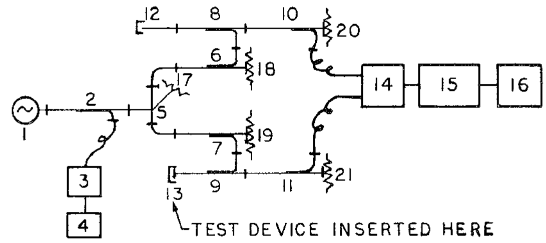

Fig. 6. Experimental arrangement for the measurement of phase shift per period and the electric field along the axis of the periodic structure: (1) microwave sweep oscillator (8 to 12.4 GHz), (2) 20-dB directional coupler, (3) wavemeter, (4) power meter, (5) magic tee, (6), (7) 3-dB directional coupler, (8)–(11) 10-dB directional coupler, (12) short circuit plunger, (13) shorting plate, (14) frequency converter, (15) network analyzer, (16) polar display, and (17)–(21) matched load.

VII. Experiment

The propagation constant of the periodic structures under study has been experimentally determined by measuring the resonant frequencies of a short-circuited length of the periodic structure having sufficient whole number of periods [17]. The shorting plate is placed at the plane of symmetry. At resonance, $\beta_0 p=q\pi/s$ where s is the number of periods and q is an integer. The values of q may be determined by moving a dielectric bead axially along the structure and counting the numbers of zero perturbations of the field.

For determining the nature of variation of the electric field along the axial direction, i.e., to obtain the variation of U versus u, the amount of field perturbation (proportional to E^2) as the bead position is varied axially has been found in a manner as suggested by Aikin [18]. The experimental data agree closely with the theoretical plot stated under Section III-B.

The experimental arrangement for the measurement of β_0 at various values of ω as well as for finding the variation of U (i.e., $|E|$) with u is shown in Fig. 6. It has been possible to achieve quick and accurate results by using a sweep oscillator and a network analyzer in the above experiment.

The experimental results of $\beta_0 p$ versus f have been obtained for a number of cases showing good agreement with analysis; typical experimental results are presented in Fig. 2. The experimental results on the variation of the electric field along the axis for several values of c have been found to agree with the corresponding calculated values.

VIII. Conclusion

The propagation constant of a rectangular waveguide with sinusoidal width modulation has been determined first by using field theory and employing 1) numerical analysis and, 2) the Hill's equation. In a second approach, circuit theory was applied.

The solution of the wave equation for the propagating wave within the structure possesses the characteristic exponent which retains all the properties of the propagation constant of the structure. Numerical analysis has been employed for the solution of the wave equation and then the exponent was evaluated. The same has also been obtained by finding the determinant of the Hill's equation. Results of both these methods are in good agreement with experimental data. In the former method, the final or true value of the propagation constant is obtained, whereas in the latter case the value is determined within the restricted range of 0–2π radians. Further, the accuracy of the numerical analysis depends on the order of the Runge–Kutta method employed. Here a fourth-order Runge–Kutta method which is of sufficiently high precision has been used. The accuracy of the analysis based on the Hill's equation, however, depends on the order of the Hill's determinant considered. In the present investigation, a 51×51 order of determinant has been used which is accurate to five decimal places.

Though the theoretical development in the above two methods of analysis is subject to constraint on the physical dimensions of the structure, i.e., $c\ll 1$, it has been found that the experimental results agree with the calculated values even for a large value of c up to 0.25. Even when the value of $c=1.25$, it was observed that though the theory failed to predict the position and the width of the stopbands, the nature of variation of the propagation constant with frequency inside the passband was predictable.

In the circuit point of view, the above physical constraint is totally absent and the accuracy of the theoretical results increases with the increase of the order of approximation. Further, the accuracy of this analysis may be improved by considering the junction susceptance at each discontinuity. This has been verified by the authors but not reported here.

The study shows that the structure supports a fast wave. It is characterized by stopbands whose widths decrease with increasing frequency and finally vanish. The distortion in the magnetic field lines for the TE_{01} mode due to width modulation has been clearly brought out.

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